On Strong Ellipticity 
and the Legendre-Hadamard Condition

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1. Introduction

The purpose of this paper is twofold. First, we derive conditions characterizing all fourth-order tensors that satisfy the strong ellipticity condition or the Legendre-Hadamard condition. Secondly, we establish the sufficiency of the Legendre-Hadamard condition for the non-negativity of the second variation in non-linear elasticity in a class of variations that satisfy a certain type of boundary condition.

In elasticity theory, the elasticity tensor \( A \) satisfies the Legendre-Hadamard condition if

\[
(a \otimes b) \cdot A(a \otimes b) \geq 0
\]

for all vectors \( a \) and \( b \), and satisfies the strong ellipticity condition if the strict version of the above inequality holds for all non-zero vectors \( a \) and \( b \). These conditions play important roles in the study of stability and wave propagation in elasticity theory. Accordingly, it is of interest to characterize, in a general way, all tensors that satisfy these conditions. By defining an appropriate linear transformation for fourth-order tensors, we show that whether a fourth-order tensor \( A \) satisfies these conditions is completely determined by the projection of its transform on a 36-dimensional subspace of the space of fourth-order tensors. Furthermore, it is found that \( A \) is strongly elliptic if and only if the associated projection maps non-zero, positive semi-definite, symmetric second-order tensors to positive-definite, symmetric tensors, and that \( A \) satisfies the Legendre-Hadamard condition if and only if the associated projection maps positive semi-definite, symmetric tensors to positive semi-definite, symmetric tensors.

VAN Hove [2] has shown, by using Fourier transforms, that if \( A \) is constant in a domain \( \Omega \), the Legendre-Hadamard condition is sufficient for the inequality

\[
\int_\Omega \nabla u \cdot A(\nabla u) \geq 0
\] (1)
to hold for all smooth functions $u$ that vanish on $\partial \Omega$. In this paper, we show that if $A$ is the elasticity tensor for an isotropic material, then the above statement holds for all smooth functions $u$ that satisfy

$$\left( \int_{\Omega} u \otimes \nabla u \otimes n \right) [W] = 0$$

for any skew tensor $W$, where $n$ is the outward unit normal to $\partial \Omega$. The last equation defines a class of variations that is much richer than that consisting of functions that vanish on $\partial \Omega$. As an application of this result, the Legendre-Hadamard condition implies the so-called infinitesimal stability\(^1\) for a homogeneous deformation of an isotropic elastic body subject to a class of boundary conditions compatible with the last equation. The proof of our results is purely algebraic, without using the Fourier transform. The basic idea of the proof is a natural one. Rewriting the left-hand side of (1) in the form

$$A \cdot \left( \int_{\Omega} \nabla u \otimes \nabla u \right),$$

we show that the essential part of this integral can be represented as a sum of tensor products of second-order tensors of rank-one.

2. Preliminaries

We denote by Lin the space of all linear transformations (tensors) from $\mathbb{R}^3$ into $\mathbb{R}^3$, by Sym the subspace of Lin consisting of all symmetric tensors, and by Skw the subspace of Lin consisting of all skew tensors. The following decomposition of Lin is well-known:

$$\text{Lin} = \text{Sym} \oplus \text{Skw}.$$  

Given an orthonormal basis $\{e_i\}$ of $\mathbb{R}^3$, we can form an orthonormal basis $\{E_{ij}\}$ of Lin by\(^2\)

$$E_{ij} = e_i \otimes e_j \text{ (no sum)},$$

$$E_{ij} = \frac{1}{\sqrt{2}} \left( e_i \otimes e_j + e_j \otimes e_i \right) \quad i = j \text{ (no sum)},$$

$$e_{ij} = \begin{cases} 1 & \text{if } \{i,j\} = \{1, 2\}, \{2, 3\}, \text{ or } \{3, 1\} \\ -1 & \text{otherwise} \end{cases}$$

Let

$$\mathcal{M} = \{(i,j): i = j \text{ or } e_{ij} = 1\}, \quad \mathcal{W} = \{(i,j): e_{ij} = -1\}.$$

It is then obvious that the set $\{E_{ij}: (i,j) \in \mathcal{M}\}$ forms an orthonormal basis of Sym and $\{E_{ij}: (i,j) \in \mathcal{W}\}$ an orthonormal basis of Skw.

\(^1\) See Truesdell & Noll [1, Section 68].

\(^2\) In this work, indices range from 1 to 3, unless otherwise stated.
It is well-known that a symmetric tensor $S$ has a spectral decomposition
\[ S = \sum \lambda_i m_i \otimes m_i, \quad |m_i| = 1, \]  
(3)
where $\lambda_i$ are (real) eigenvalues of $S$. We denote by $\text{Sym}_+$ the subset of $\text{Sym}$ consisting of all symmetric tensors that are positive semi-definite, i.e., for which $\lambda_i \geq 0$ in (3), and by $\text{Sym}^-$ all symmetric tensors that are positive-definite, i.e., for which $\lambda_i > 0$ in (3).

Let $\mathcal{A}$ denote the space of all linear transformations from $\text{Lin}$ into itself, associated with the inner product defined in the usual manner, e.g., in components with respect to $\{e_i\}$,
\[ A \cdot B = \sum_{i,j,k,l} A_{ijkl} B_{ijkl}. \]

It is an easy matter to show that $\mathcal{A}$ has the following decomposition
\[ \mathcal{A} = (\text{Sym} \otimes \text{Sym}) \oplus (\text{Sym} \otimes \text{Skw}) \oplus (\text{Skw} \otimes \text{Sym}) \oplus (\text{Skw} \otimes \text{Skw}) \]  
(4)
where
\[ \text{Sym} \otimes \text{Sym} = \left\{ A \in \mathcal{A} : A = \sum_{i,j,k,l} a_{ijkl} E_{ij} \otimes E_{kl}, a_{ijkl} \in \mathbb{R}, (i,j) \in \mathcal{M}, (k,l) \in \mathcal{M} \right\}, \]

and the other three components on the right-hand side of (4) are defined similarly.

We define a subspace $\mathcal{G}$ of $\mathcal{A}$ by
\[ \mathcal{G} = \left\{ G \in \mathcal{A} : G = \sum_{i=1}^{n} g_i a_i \otimes a_i \otimes b_i \otimes b_i, g_i \in \mathbb{R}, a_i, b_i \in \mathbb{R}^3, \text{\ n a positive integer} \right\}. \]

We denote by $P(A)$ the orthogonal projection of $A \in \mathcal{A}$ on $\mathcal{G}$. The following proposition identifies $\mathcal{G}$ with $\text{Sym} \otimes \text{Sym}$, and therefore makes it clear how to find $P(A)$ for a given $A$.

**Proposition 1.** $\mathcal{G} = \text{Sym} \otimes \text{Sym}$.

**Proof.** We observe from (4) and (5) that the set $\text{Sym} \otimes \text{Sym}$ consists of all fourth-order tensors that map $\text{Sym}$ into $\text{Sym}$ and map $\text{Skw}$ to $\{0\}$. Yet, each $G$ in $\mathcal{G}$ has such a property. Hence, $\mathcal{G} \subset \text{Sym} \otimes \text{Sym}$. To show that $\text{Sym} \otimes \text{Sym} \subset \mathcal{G}$, we note that $E_{ij}$ and $E_{kl}$ in (5) are symmetric and have a spectral decomposition.

Let $\mathcal{G}_+$ be the subset of $\mathcal{G}$ such that each $G$ in $\mathcal{G}_+$ has a representation similar to those in $\mathcal{G}$ but with $g_i \geq 0, \ i = 1, 2, \ldots, n$.

For an $A \in \mathcal{A}$, we define $A'$, the "2–3 transpose" of $A$, by
\[ A'[u \otimes v] w = A[w \otimes v] u \quad \text{for all } u, v, w \in \mathbb{R}^3. \]
In components with respect to \( \{ e_i \} \),
\[
A_{jkl}^i = A_{ijkl}.
\]

We write
\[
\text{Lin}^- = \{ F \in \text{Lin}: \det F > 0 \},
\]
\[
\text{Orth}^+ = \{ Q \in \text{Lin}^-: Q^TQ = I \},
\]
where \( I \) is the identity tensor, and a superimposed \( T \) denotes the usual transpose.

3. The Constitutive Relation, Strong Ellipticity

We consider a homogeneous elastic body that occupies a bounded regular domain \( \Omega \subset \mathbb{R}^3 \) in a fixed reference configuration. A deformation of the body is represented by an invertible function \( \mathbf{x} \in C^1(\Omega; \mathbb{R}^3) \), such that \( \nabla \mathbf{x} \in \text{Lin}^- \). We denote by \( F \) the deformation gradient \( \nabla \mathbf{x} \), which admits the polar decomposition
\[
F = RU,
\]
where \( R \in \text{Orth}^- \) and \( U \in \text{Sym}^- \).

The body is associated with a response function \( S \in C^1(\text{Lin}^-; \text{Lin}) \). For a deformation \( \mathbf{x} \), the Piola-Kirchhoff stress is given by \( S(\nabla \mathbf{x}) \). By the principle of frame-indifference, the response function satisfies
\[
S(QF) = QS(F) \quad \text{for all } Q \in \text{Orth}^+, F \in \text{Lin}^-.
\]

The body is isotropic if
\[
S(FQ) = S(F) Q \quad \text{for all } Q \in \text{Orth}^+, F \in \text{Lin}^-.
\]
It is well-known\(^3\) that (7) and (8) imply the following representation of the response function:
\[
S(F) = s_1 F + s_2 FF^T F + s_3 F^{-T},
\]
where \( s_i \) are scalar functions of the principal invariants of \( U \), and \( F^{-T} = (F^T)^{-1} \).

The elasticity tensor at the deformation gradient \( F \) is defined by
\[
A(F) = \frac{dS}{dF}(F).
\]
The elasticity tensor is said to be strongly elliptic at \( F \) if
\[
(a \otimes b) \cdot A(F) [a \otimes b] > 0 \quad \text{for all } a, b \in \mathbb{R}^3, a \neq 0, b \neq 0.
\]

\(^3\) See, for example, TRUESDELL & NOLL [1].
Proposition 2. The elasticity tensor is strongly elliptic at \( F \) if and only if
\[
P\langle A(F)^{\gamma} \rangle \cdot G > 0 \quad \text{for all } G \in \mathcal{G}_+ \setminus \{O\}.
\]

Proof. Let \( G \in \mathcal{G}_+ \setminus \{O\} \) be given with
\[
G = \sum_{i=1}^{n} g_i a_i \otimes a_i \otimes b_i \otimes b_i, \quad g_i > 0, \quad a_i = 0, \quad b_i = 0.
\]

Since \( P(G) = G \), we have
\[
P\langle A(F)^{\gamma} \rangle \cdot G = A(F)^{\gamma} \cdot G = A(F) \cdot G - \sum_{i=1}^{n} g_i (a_i \otimes b_i) \cdot A(F) [a_i \otimes b_i].
\]

The last expression is positive if \( A(F) \) is strongly elliptic, which proves necessity. To show sufficiency, we note that
\[
(a \otimes b \otimes a \otimes b)^{\gamma} \in \mathcal{G}_+ \quad \text{for all } a, b \in \mathbb{R}^3,
\]
and that
\[
(a \otimes b) \cdot A(F) [a \otimes b] = P\langle A(F)^{\gamma} \rangle \cdot (a \otimes b \otimes a \otimes b)^{\gamma}. \quad \square
\]

Proposition 2 shows, among other things, that the strong ellipticity of the elasticity tensor \( A \) is completely determined by the projection of \( A' \) on \( \mathcal{G} \). In general, \( A \) has 81 components with respect to the basis \( \{E_{ij} \otimes E_{kl}\} \). Only 36 of them play a role in the strong ellipticity condition for \( A \).

Theorem 1. Suppose that \( A \in \mathcal{G} \). Then
\[
A \cdot G > 0 \quad \text{for all } G \in \mathcal{G}_+ \setminus \{O\}
\]
if and only if \( A \) maps \( \text{Sym}_+ \setminus \{O\} \) into \( \text{Sym}^- \).

Proof. Necessity. Let \( S \in \text{Sym}_+ \setminus \{O\} \) be given with
\[
S = \sum_{i=1}^{n} \lambda_i m_i \otimes m_i, \quad \lambda_i \geq 0, \quad \max \{\lambda_i\} > 0, \quad |m_i| = 1.
\]
Then \( A[S] \in \text{Sym} \) since \( A \in \mathcal{G} \). Moreover, for any \( v \in \mathbb{R}^3 \), \( v = 0 \),
\[
v \cdot A[S] v = A \cdot \left( \sum_{i=1}^{n} \lambda_i v \otimes v \otimes m_i \otimes m_i \right).
\]
By hypothesis, the right-hand side is positive since the fourth-order tensor in the parenthesis belongs to \( \mathcal{G}_+ \setminus \{O\} \).

Sufficiency. Let \( G \in \mathcal{G}_+ \setminus \{O\} \) be given with a representation (10). Then
\[
A \cdot G = \sum_{i=1}^{n} g_i (a_i \otimes a_i) \cdot A[b_i \otimes b_i].
\]
Each term on the right-hand side is positive by hypothesis. \( \square \)
Corollary 1. A necessary and sufficient condition for the elasticity tensor $A$ to be strongly elliptic is that the projection of $A'$ on $\mathcal{G}$ map each non-zero, positive semi-definite, symmetric tensor to a positive-definite, symmetric tensor.

Proposition 3. The elasticity tensor is strongly elliptic at $F$ if and only if it is strongly elliptic at $U$.

Proof. Differentiating (7) with respect to $F$, setting $Q = R^T$ and using (6), we find that

$$H \cdot A(F)[H] = (R^T H) \cdot A(U)[R^T H] \quad \text{for all } H \in \text{Lin}. \quad (11)$$

The desired result then follows. □

The elasticity tensor is said to satisfy the Legendre-Hadamard condition at $F$ if

$$(a \otimes b) \cdot A(F)[a \otimes b] \geq 0 \quad \text{for all } a, b \in \mathbb{R}^3.$$

Since the above inequality is the non-strict version of the inequality appearing in the definition of strong ellipticity, all results we have thus far concerning strong ellipticity can be made appropriate for the Legendre-Hadamard condition as follows.

Proposition 4. The elasticity tensor satisfies the Legendre-Hadamard condition at $F$ if and only if

$$P(A(F))' \cdot G \geq 0 \quad \text{for all } G \in \mathcal{G}_+. $$

Theorem 2. Suppose that $A \in \mathcal{G}$. Then

$$A \cdot G \geq 0 \quad \text{for all } G \in \mathcal{G}_-$$

if and only if $A$ maps $\text{Sym}_+$ into itself.

Corollary 2. A necessary and sufficient condition for the elasticity tensor $A$ to satisfy the Legendre-Hadamard condition is that the projection of $A'$ on $\mathcal{G}$ maps each positive semi-definite symmetric tensor to a positive semi-definite symmetric tensor.

Proposition 5. The elasticity tensor satisfies the Legendre-Hadamard condition at $F$ if and only if it does so at $U$.

4. Homogeneous Deformations of an Isotropic Body

A deformation $x$ is homogeneous if $\nabla x$ is constant in $\Omega$. In this case, the values of the response function and the elasticity tensor are constant as well since we are considering a homogeneous body.
In the study of stability and wave propagation in elasticity theory, it is of interest to find conditions on the elasticity tensor such that the following inequality holds:

$$\int_B \nabla u \cdot A(F)[\nabla u] \geq 0 \quad \text{for all } u \in C^4(\Omega; \mathbb{R}^3).$$

(12)

If the response function is derived from a strain-energy function, inequality (12) stands as the second variation condition associated with the minimization of the integral of the strain-energy function over $\Omega$.

It is well-known\(^4\) that a necessary condition for (12) to hold is that $A(F)$ satisfy the Legendre-Hadamard condition. This condition is not sufficient in general. However, van Hove [2] showed that if $F$ is constant in $\Omega$, then the Legendre-Hadamard condition is sufficient for (12) to hold provided that $u|_{\partial \Omega} = 0$. The technique employed by van Hove uses the Fourier transform to convert the left-hand side of (12) to an integral in which the integrand consists of quadratic forms of $A(F)$ with rank-one tensors.

In this work, we shall prove, by using a totally different approach, a strengthened version of van Hove's theorem for isotropic bodies. Precisely, we shall show that if $F$ is constant in $\Omega$, if the response function is given by (9), and if $A(F)$ satisfies Legendre-Hadamard condition, then inequality (12) holds for all $u \in C^4(\Omega; \mathbb{R}^3)$ that satisfy

$$\left( \int_{\partial \Omega} u \otimes \nabla u \otimes n \right)[W] = 0 \quad \text{for all } W \in \text{Skw},$$

(13)

where $n$ is the outward unit normal to $\partial \Omega$. That the condition (13) is weaker than $u|_{\partial \Omega} = 0$ is readily illustrated by the example

$$u(X) = \varphi(X) \mathbf{a},$$

$\mathbf{a}$ being a non-zero vector and $\varphi \in C^4(\Omega; \mathbb{R})$.

Henceforth, we take $E_{ij}$ defined in (9) to be such that $e_1, e_2, e_3$ are three orthonormal eigenvectors of $U$. The elasticity tensor $A$ for an isotropic body can be found by differentiating (9) with respect to $F$. In particular, we have

$$E_{ij} : P(A(U)^r)[E_{kl}] = 0 \quad \text{for } \epsilon_{ij} = 1 \text{ or } \epsilon_{kl} = 1, (i, j) = (k, l),$$

$$E_{ij} : P(A(U)^r)[E_{ij}] = E_{ij} : P(A(U)^r)[E_{ij}] \text{ (no sum)}.$$

This shows that $P(A(U)^r)$ has at most 12 non-zero components with respect to $\{E_{ij} \otimes E_{kl}\}$, among which there are three identical pairs. Hence, we can write

$$P'(A(U)^r) = \sum_{(i,j) \neq (i',j')} a_{ij} E_{ij} \otimes E_{ij} - \sum_{i \neq j} a'_{ij} E_{ii} \otimes E_{jj},$$

(14)

where $a_{ij}$ and $a'_{ij}$ are scalar functions of the principal invariants of $U$, with $a'_{ij} = a_{ij}$.

\(^4\) See Truesdell & Noll [1, Section 68].
Lemma 1. Suppose that $u \in C^2(\Omega; \mathbb{R}^3)$ satisfies (13). Then
\[
\left( \int_{\Omega} L \nabla u \otimes L \nabla u \right) \in \mathcal{H} \quad \text{for all } L \in \text{Lin}.
\]

Proof. Let an $L \in \text{Lin}$ be given. It suffices to show that
\[
\left( \int_{\Omega} L \nabla u \otimes L \nabla u \right) [S] \in \text{Sym} \quad \text{for all } S \in \text{Sym},
\]
\[
\left( \int_{\Omega} L \nabla u \otimes L \nabla u \right) [W] = 0 \quad \text{for all } W \in \text{Skw}.
\]

Let an $S \in \text{Sym}$ be given with the spectral decomposition (3). Then
\[
\left( \int_{\Omega} L \nabla u \otimes L \nabla u \right) [S] = \sum_i \lambda_i \int_{\Omega} (L \nabla u m_i) \otimes (L \nabla u m_i).
\]

The right-hand side is obviously symmetric. Moreover, for $W \in \text{Skw}$, we use the divergence theorem to find that
\[
\left( \int_{\Omega} L \nabla u \otimes L \nabla u \right) [W] = -\left( \int_{\partial \Omega} L u \nabla u \nabla u \right) [W].
\]

The right-hand side vanishes by (13). □

Lemma 2. Suppose that $u \in C^2(\Omega; \mathbb{R}^3)$ satisfies (13). Then
\[
\left( \int_{\Omega} L \nabla u \otimes L \nabla u \right) [a \otimes b] c = \left( \int_{\Omega} L \nabla u \otimes L \nabla u \right) [a \otimes c] b
\]
for all $a, b, c \in \mathbb{R}^3$, and $L \in \text{Lin}$.

Proof. By using the divergence theorem, we find that
\[
\left( \int_{\Omega} L \nabla u \otimes L \nabla u \right) [a \otimes b] c = \left( \int_{\Omega} L \nabla u \otimes L \nabla u \right) [a \otimes c] b
\]
\[
+ \left( \int_{\partial \Omega} L u \nabla u \nabla u \right) [b \otimes c - c \otimes b] a.
\]

The conclusion then follows from (13). □

Lemma 3. Suppose that $u \in C^2(\Omega; \mathbb{R}^3)$ satisfies (13). Define
\[
g_{ij} = (E_i \otimes E_j) \cdot \left( \int_{\Omega} R^T \nabla u \otimes R^T \nabla u \right) \quad (i, j) \in \mathcal{M}, \quad (\text{no sum}),
\]
\[
g_{ij} = \frac{1}{2} (E_i \otimes E_j + E_j \otimes E_i) \cdot \left( \int_{\Omega} R^T \nabla u \otimes R^T \nabla u \right) \quad i \neq j \quad (\text{no sum}).
\]

(15)
Then
\[ |g_0| \leq 2g' \quad \text{if } e_H = 1, \quad (16) \]
and the matrix
\[ (G) = \begin{pmatrix} g_{11} & 2g_{12} & 2g_{13} \\ 2g_{12} & g_{22} & g_{23} \\ 2g_{13} & g_{23} & g_{33} \end{pmatrix} \quad (17) \]
is positive semi-definite.

**Proof.** By Lemma 2 and the hypothesis, we have
\[
g_u = \int_D \left( e_i \cdot R^T \nabla u e_i \right)^2 \quad \text{(no sum)},
\]
\[
g_u = 2 \int_D \left( e_i \cdot R^T \nabla u e_i \right) \left( e_j \cdot R^T \nabla u e_j \right)
= 2 \int_D \left( e_i \cdot R^T \nabla u e_j \right) \left( e_j \cdot R^T \nabla u e_i \right) \quad \text{for } e_{ii} = 1 \quad \text{(no sum)},
\]
\[
g' = \frac{1}{4} \int_D \left[ (e_i \cdot R^T \nabla u e_i)^2 + (e_j \cdot R^T \nabla u e_j)^2 \right].
\]
Inequality (16) follows immediately. To show that the matrix \( G \) is positive semi-definite, let \( a_1, a_2, a_3 \in \mathbb{R} \) be given and observe that
\[
\begin{align*}
g_{11}a_1^2 + g_{22}a_2^2 + g_{33}a_3^2 + g_{12}a_1a_2 + g_{23}a_2a_3 + g_{31}a_3a_1 \\
= \int_D \left( \sum_i a_i e_i \cdot R^T \nabla u e_i \right)^2 \geq 0. \quad \square
\end{align*}
\]

**Lemma 4.** For a \( G \in \text{Sym}_n \), there exist \( v_1, v_2, \ldots, v_n \in \mathbb{R}^3 \), \( n \) being a positive integer, such that
\[
\left| \sum_{k=1}^n (e_i \cdot v_k) (e_j \cdot v_k) \right| = \sum_{k=1}^n (e_i \cdot v_k) (e_j \cdot v_k), \quad (18)
\]
\[
G = \sum_{k=1}^n v_k \otimes v_k. \quad (19)
\]

**Proof.** \( G \) has a spectral decomposition (3). We assume that the eigenvalues of \( G \) are ordered as
\[
\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 0.
\]
We then have
\[
G = \lambda_3 I + \sum_{l=1}^2 \left( \lambda_l - \lambda_3 \right) m_l \otimes m_l.
\]
It then suffices to prove the lemma for those \( G \) of the form
\[
G = a \otimes a + b \otimes b, \quad a, b \in \mathbb{R}^3, \quad a \cdot b = 0.
\]

Let such a pair \( a \) and \( b \) be given. A straightforward calculation shows that we can find nonnegative numbers \( \alpha, \beta, \alpha_i, \) and \( \beta_i \) such that
\[
a \otimes a + b \otimes b = \alpha a \otimes a + \beta b \otimes b + \sum_i \left[ \alpha_i e_i \otimes e_i + \beta_i [e_i \times (a \times b)] \otimes [e_i \times (a \times b)] \right],
\]
and that each component of the right-hand side is the sum of eight numbers of same sign (including zero). \( \square \)

**Theorem 3.** Suppose that \( F \) is constant in \( \Omega \), that \( S(F) \) satisfies the isotropy condition (8), and that \( A(F) \) satisfies the Legendre-Hadamard condition. Then
\[
\int_\Omega \nabla u \cdot A(F) [\nabla u] \geq 0
\]
for all \( u \in C^2(\Omega; \mathbb{R}^3) \) that satisfy (13).

**Proof.** By (11), (14), Lemma 1, and the fact that \( F \), and therefore \( U \), is constant in \( \Omega \), we find that
\[
\int_\Omega \nabla u \cdot A(F) [\nabla u] = \int_\Omega (R^T \nabla u) \cdot A(U) [R^T \nabla u]
\]
\[
= A(U)^T \cdot \left( \int_\Omega R^T \nabla u \otimes R^T \nabla u \right)^T
\]
\[
= P(A(U)^T) \cdot \left( \int_\Omega R^T \nabla u \otimes R^T \nabla u \right)^T
\]
\[
= P(A(U)^T) \cdot G
\]
where
\[
G = \sum_{i,j \in \mathcal{A}} g_{ij} E_{ij} \otimes E_{ij} + \sum_{i \neq j} g'_{ij} E_{ii} \otimes E_{jj},
\]
(20)
g\(_{ij}\) and \(g'_{ij}\) being defined by (15). By Propositions 4 and 5, it now suffices to show that \( G \) defined by (20) belongs to \( \mathcal{G}_+ \). To do so, we note from Lemmas 3 and 4 that the matrix \( (G) \) defined by (17) is positive semi-definite and therefore has a representation (19) with (18) satisfied, that is, there exist \( v_{ki} \in \mathbb{R}, \ k = 1, 2, \ldots, n, \ i = 1, 2, 3, \) such that
\[
\left| \sum_{k=1}^n w_{ki} v_{ki} \right| = \sum_{k=1}^n |v_{ki} v_{ki}|,
\]
(21)
and
\[
g_{ii} = \sum_{k=1}^n v_{ki}^2, \quad \text{(no sum on } i),
\]
\[
g_{ij} = 2 \sum_{k=1}^n v_{ki} v_{kj}, \quad \text{for } i \neq j.
\]
We define
\[ v_k = \sum_i |v_{ki}|^{1/2} e_i, \]
\[ \hat{v}_k = \sum_i |v_{ki}|^{1/2} \text{sgn}(v_{ki}) e_i \quad \text{(no sum)}, \]
\[ I_0 = I, \]
\[ I_i = I - 2e_i \otimes e_i \quad \text{(no sum)}. \]

A straightforward calculation, with the aid of (17), (20) and (21), then yields
\[ G = \frac{1}{4} \sum_{k=1}^3 \sum_{j=0}^3 I_k v_k \otimes I_j \hat{v}_k \otimes I_j \hat{v}_k + \sum_{j=0} \left( g_{ij} - \frac{1}{4} |g_{ij}| \right) E_{ii} \otimes E_{ii}. \]

By (16), the right-hand side belongs to \( \mathcal{G}_+ \), which completes the proof. \( \square \)

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